Scaling theory

What does the functional form of the correlation function teach us?

$$c(r,t) = C_0 \frac{1}{r^{d-2}} e^{-r/\xi} \qquad \text{with} \qquad \xi = a|t|^{-\nu} \qquad \nu = 1/2 \text{ in mean-field}$$

Let's say that we move away from the system, with our ruler in front of us and let's say that I look at two points separated by ξ. Let's mark two points on the ruler, at distance ξ. As I move away, the two points on the ruler will not coincide with the two points in the system, but they will be further apart.

Essentially, the original distances look shorter

$$r' = \frac{r}{b}$$

$b > 1$ is the scaling factor to translate measures on the system to measures taken from far away

Then

$$c(r,t) = c_0 \frac{e^{-r/b\left(\frac{1}{b}\xi\right)}}{\left(r/b\right)^{d-2} b^{d-2}} = c_0 \frac{e^{-r'/(\xi/b)}}{(r')^{d-2} b^{d-2}}$$

obviously, also the correlation length has shrinked : $\xi' = \xi/b$

but we recall that $\xi = a|t|^{-1/2}$ (in general $|t|^{-1/\nu}$)

Then $\frac{\xi}{b} = a|b^2 t|^{-1/2} = a|t'|^{-1/2}$ $t' = b^2 t$

we can thus rewrite

$$c(r,t) = b^{-(d-2)} c(r',t')$$
$$= b^{-(d-2)} c\left(\frac{r}{b}, b^2 t\right)$$

the correlation function is a homogeneous function of its arguments

Note: homogeneous function : $f(x) = \lambda^{\alpha} f(\lambda x)$

Physically, what does it mean?

As we move farther away, the system behaves in the same way as if it had a different temperature:

$$t' = b^2 t > t$$

but t is the offset from criticality: T moves away from T_c.

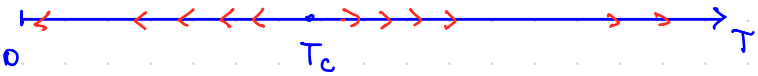
Is this intuitively correct?

ξ is the correlation length: for exemple below T_c we can expect the system to be mostly ordered in one direction, with bubbles of size ξ of the opposit sign.

Moving away, the bubbles will look smaller and smaller until they disappear, the system looks completely ordered : $T=0$

By converse, if $T > T_c$, the system is globally disordered, but with some ordered bubbles (in equal number + and -) of size ξ. Moving away, those bubbles become smaller and smaller, as if the system was disordered down to its smaller scale : $T = \infty$

We can graphically represent this in this way:



rescaling the system, it goes to its "archetypical" states : the fully ordered at $T=0$ and fully disordered at $T=\infty$.
They are the stable fixed points of the rescaling procedure.

What about T_c? it is an unstable fixed point.
But it is a fixed point of the rescaling, where $\xi = \infty$

What does $\xi = |t|^{-1/2} = \infty$ mean?

1) There is no typical lengthscale in the system

2) The system looks the same at all scales !!!

There will be correlated bubbles at all scales!

This is the concept of $\boxed{\text{scale invariance}}$

Note: also $T=0$ and $T=\infty$ are scale invariant, because the correlation length $\xi = 0 \Rightarrow \xi/b = 0$: it does not change upon rescaling.

Before moving on, just a quick historical note on critical exponents.

Critical exponents had been observed and measured well before the theoretical study we are going through had been introduced. Using thermodynamics, it had been derived that critical exponents had to satisfy inequality relations:

Rushbrooke inequality: $\alpha + 2\beta + \gamma \geq 2$

Griffiths inequality: $\alpha + \beta(1+\delta) \geq 2$

Josephson $(\underbrace{\text{hyperscaling}}_{\text{because it involves } d})$ relation: $2 - \alpha \geq d\nu$

Fisher's inequality: $\gamma \geq (2-\eta)\nu$

and a few others.

What was striking was that, experimentally and in the theoretical cases where they were known (e.g. Ising 2d), the inequalities were respected as equalities, and there was no theory for this.

These considerations set the stage for the next step in scaling theory: the structure of the free energy.

We have seen that, at the phase transition, the free energy must be non-analytic. This means that it can be written as

$$f(T,h) = f_{\text{non-analytic}}(T,h) + f_{\text{analytic}}(T,h)$$

and the only part that concerns us is the non-analytic one.

From now on, we drop the "non-analytic" label. In what follows $f(T,h)$ will be just the non-analytic part.

The scaling relations at the critical point, namely

$$m = a|t|^{\beta} \qquad C = C_0 |t|^{-\alpha}$$

$$\chi = \chi_0 |t|^{-\gamma} \qquad m = m_0 h^{1/\delta}$$

are homogeneous relations. For exemple

$$m(t) = a|t^{\beta}| = a\lambda^{\beta}\left(\left|\frac{t}{\lambda}\right|\right)^{\beta} = \lambda^{\beta} m\left(\frac{t}{\lambda}\right)$$

But all those quantities are derivatives of the free energy $f(T,h)$.

This means that close to the critical point the free energy must be a homogeneous function

$$f(t,h) = \lambda^{\varepsilon} f(t\lambda^{y_t}, h\lambda^{y_h})$$

$\uparrow\ \nearrow$ we use reduced, adimensional variables

But we have seen from the correlation length that a natural scaling variable for the temperature is the length rescaling factor b

$$t\, b^{y_t}$$

($y_t = 2$ in mean field, $1/\nu$ more generally)

The scaling of the energy is thus

$$f(t,h) = b^{\varepsilon} f(t b^{y_t}, h b^{y_h})$$

and we still have to decide about ε.

Lower-case f is the free-energy <u>per unit volume</u> !

The rescaling factor, we said earlier, is associated to a unit of length that is b times larger than before. A unit of volume after rescaling is thus b^d larger than before.

Since energy is extensive, this rescaled unit of volume contains b^d more energy than before rescaling.

The two energies, before and after rescaling, are thus the same if the one after rescaling is divided by b^d

$$\boxed{f(t,h) = b^{-d} f(t b^{y_t}, h b^{y_h})}$$

This is the scaling hypothesis as formulated by Widom and Fisher (and later developed by Kadanoff) in the 1960's.

__Important__: it is an hypothesis. Very reasonable, but the proof came only later.

Consequences of the scaling hypothesis: all critical exponents can be derived from y_t and y_h:

$$m = \frac{\partial f}{\partial h} \Rightarrow \quad \partial_h f(t,h) = \partial_h \left[b^{-d} f(t b^{y_t}, h b^{y_h}) \right] =$$

$$= b^{-d+y_h} f(t b^{y_t}, h b^{y_h})$$

The spontaneous magnetization is calculated for $h=0$

$$\partial_h f(t,0) = b^{-d+y_h} \partial_h f(t b^{y_t}, 0)$$

The scaling factor is arbitrary: we choose $b = |t|^{-1/y_t}$

$$\Rightarrow \quad \partial_h f(t,0) = |t|^{\frac{d-y_h}{y_t}} \partial_h f(1,0)$$

$$\Rightarrow \quad \beta = \frac{d-y_h}{y_t}$$

Working along the same ideas leads to the proof that the inequalities are actually equalities.